$\mathcal{O}(1)$:

$$\frac{\partial F_0}{\partial V} = \nu_0 \left(\delta(V-V_r) - \delta(V-V_t) \right)$$

$$F_0 = -\frac{\sigma^2}{2\tau} \frac{\partial P_0}{\partial V} + (V-\mu_0) P_0$$

let $F_0 = \nu_0 f_0$

$P_0 = \nu_0 p_0$

$$\frac{\partial f_0}{\partial V} = \delta(V-V_r) - \delta(V-V_t)$$

$$-\frac{\partial p_0}{\partial V} = \frac{2}{\sigma^2}(V-\mu_0) p_0 + \frac{2\tau}{\sigma^2} f_0 \qquad (9)$$

Now, Eq. 9 is of form

$$-\frac{dp_0}{dV} = G p_0 + H \quad , \quad G = \frac{2}{\sigma^2}(V-\mu_0) \; , \; H = \frac{2\tau}{\sigma^2} f_0(V)$$

integrate.

$$\frac{dp_0}{dV} + G p_0 = -H$$

$$\frac{d}{dV}\left(p_0 \, e^{\int_{V_k}^{V} dx\, G(x)} \right) = -H e^{\int_{V_k}^{V} dx\, G(x)}$$

$$p_0(V) = p_0^k \, e^{-\int_{V_k}^{V} dx\, G(x)} - e^{-\int_{V_k}^{V} dx\, G(x)} \int_{V_k}^{V} du\, H(u)\, e^{\int_{V_k}^{u} dx\, G(x)}$$

$p_0(V_k) \equiv p_0^k$

$$p_0^{k-1} = p_0^k \, e^{+\int_{V_{k-1}}^{V_k} dx\, G(x)} + e^{\int_{V_{k-1}}^{V_k} dx\, G(x)} \int_{V_{k-1}}^{V_k} du\, H(u)\, e^{\int_{V_k}^{u} dx\, G(x)}$$

$$= p_0^k \, e^{\int_{V_{k-1}}^{V_k} dx\, G(x)} + \int_{V_{k-1}}^{V_k} du\, H(u)\, e^{\int_{V_{k-1}}^{u} dx\, G(x)}$$

Now take G, H to be constants in the integrals i.e., $\int_{V_{k-1}}^{V_k} dx\, G(V_k) = \Delta \cdot G(V_k)$, $\Delta = V_k - V_{k-1}$

$$\int_{V_{k-1}}^{u} dx\, G(V_k) = (u - V_{k-1})\, G(V_k)$$

$$\int_{V_{k-1}}^{V_k} du\, H(V_k)\, e^{(u-V_{k-1})G(V_k)} = \frac{H(V_k)}{G(V_k)} e^{G(V_k)(u-V_{k-1})}\Big|_{V_{k-1}}^{V_k}$$

$$= \frac{H(V_k)}{G(V_k)}\left(e^{\Delta \cdot G(V_k)} - 1\right)$$

$$\Rightarrow p_0^{k-1} = p^k \underbrace{e^{\Delta \cdot G^k}}_{A^k} + \frac{H^k}{G^k}\left(e^{\Delta \cdot G^k} - 1\right)$$

$$= p_0^k A^k + \frac{2\tau}{\sigma^2} f_0^k \frac{\left(e^{\Delta \cdot G^k} - 1\right) \cdot \Delta}{\Delta \cdot G^k}$$

let $B^k = \dfrac{\left(e^{\Delta \cdot G^k} - 1\right)}{\sigma^2 \Delta \cdot G^k}$ with $B^k = \dfrac{1}{\sigma^2}$ if $G^k = 0$

$$\boxed{\begin{aligned} & p_0^{k-1} = p_0^k A^k + \Delta 2\tau B^k f_0^k \\ & f_0^{k-1} = f_0^k - \delta_{k,k_r+1} \\ & p_0(V_t) = 0 \qquad N \text{ points from } V_{lb} \text{ to } V_t \\ & f_0(V_t) = 1 \\ & \text{Now, integrate backwards from } V_t \text{ to } V_{lb} \\ & V_0 = \frac{1}{\sum_{k=0}^{N} \Delta \cdot p_0^k}, \quad P_0^k = V_0 p_0^k, \quad F_0^k = V_0 f_0^k \end{aligned}}$$

$\mathcal{O}(\epsilon)$:

$$i\omega \rho_1 + \frac{\partial F_1}{\partial V} = \nu_1 \left(\delta(V-V_r) - \delta(V-V_t) \right)$$

$$-\tau F_1 = \frac{\sigma^2}{2} \frac{d\rho_1}{dV} + (V-\mu_0)\rho_1 - \mu_1 \rho_0$$

let $\rho_1 = \nu_1 \rho_\nu + \mu_1 \rho_\mu$

$F_1 = \nu_1 f_\nu + \mu_1 f_\mu$

i. $\nu_1 = 1$, $\mu_1 = 0$ (effect of rate perturbation)

$$i\omega \rho_\nu + \frac{df_\nu}{dV} = \delta(V-V_r) - \delta(V-V_t)$$

$$-\frac{df_\nu}{dV} = i\omega \rho_\nu - \delta(V-V_r) + \delta(V-V_t)$$

$$f_\nu^{k-1} = f_\nu^k + \Delta \cdot i\omega \rho_\nu^k - \delta_{k,k_r+1}$$

$$-\tau f_\nu = \frac{\sigma^2}{2} \frac{d\rho_\nu}{dV} + (V-\mu_0)\rho_\nu$$

$$-\frac{d\rho_\nu}{dV} = \frac{2}{\sigma^2}(V-\mu_0)\rho_\nu + \frac{2\tau}{\sigma^2} f_\nu$$

$$\rho_\nu^{k-1} = \rho_\nu^k A^k + \Delta \cdot 2\tau B^k f_\nu^k$$

$$f_\nu(V_t) = 1, \quad \rho_\nu(V_t) = 0$$

ii. $\nu_1 = 0$, $\mu_1 = 1$ (effect of input perturb)

$$i\omega \rho_\mu + \frac{df_\mu}{dV} = 0$$

$$f_\mu^{k-1} = f_\mu^k + \Delta \cdot i\omega \rho_\mu^k$$

$$-\tau f_\mu = \frac{\sigma^2}{2}\frac{d\rho_\mu}{\partial V} + (V-\mu_0)\rho_\mu - \rho_0$$

$$-\frac{d\rho_\mu}{dV} = \frac{2}{\sigma^2}(V-\mu_0)\rho_\mu + \frac{2\tau}{\sigma^2} f_\mu - \frac{2}{\sigma^2}\rho_0$$

$$\boxed{\rho_\mu^{k-1} = \rho_\mu^k A^k + \Delta \cdot 2\tau \left(f_\mu^k - \rho_0^k \right) B^k}$$

$$f_\mu(V_t) = \rho_\mu(V_t) = 0$$

Take a lower bound V_{lb} low enough so that the flux will be zero, then

$$F_1(V_{lb}) = \nu_1 f_\nu(V_{lb}) + \mu_1 f_\mu(V_{lb}) = 0$$

$$\Rightarrow \nu_1 = -\mu_1 \frac{f_\mu(V_{lb})}{f_\nu(V_{lb})} \qquad (11)$$

but $\mu_1 = \mu_1(\nu_1)$ so Eq. (11) is a self-consistent condition for an instability with freq. ω

Do the following exercise: consider an E-I network. Now you need to take

$$\rho_{A,1} = \nu_{A,1}\rho^\nu_{A,1} + \mu_A^E \rho^E_{A,1} + \mu_A^I \rho^I_{A,1} \quad \text{where } A = \{E, I\}$$

$$F_{A,1} = \nu_{A,1} f^\nu_{A,1} + \mu_A^E f^E_{A,1} + \mu_A^I f^I_{A,1}$$

Show that the condition for an instability is

$$\left[f^\nu_{E,1}(V_{lb}) + f^E_{E,1}(V_{lb})\right]\left[f^\nu_{I,1}(V_{lb}) + f^I_{I,1}(V_{lb})\right] - f^E_{I,1}(V_{lb})\, f^I_{E,1}(V_{lb}) = 0$$

FIN