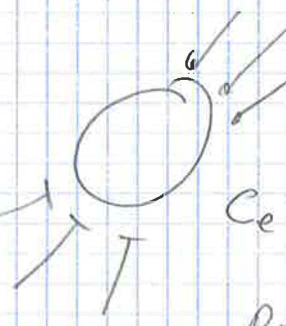
May 29th 2016

Notes for the tutorial on Meanfield Models for Networks of Spiking Neurons, ICMNS May 29th 2016. Given by Alex Roxin.

A. The Diffusion Approximation



A single cell receiving a large number C_e of excitatory and C_i of inhibitory inputs. The inputs are independent Poisson processes with rates ν_e and ν_i, respectively.

We will consider a leaky integrate-and-fire neuron (LIF)

$$\tau_m \dot{V} = -V + I(t) \quad (1)$$

$$I(t) = J_e \sum_{i=1}^{C_e} \sum_j \delta(t - t_j^i) - J_i \sum_{i=1}^{C_i} \sum_j \delta(t - t_j^i)$$

There is a slight difference in notation compared to the chapter by Renart et al which I'm following.

→ I'm taking all synapses to have the same efficacy.

The j^{th} spike of input i happens at a time t_j^i and causes a jump of amplitude J_e (or $-J_i$) in the voltage.

The dynamics of V are stochastic, so let's consider the evolution of the probability distribution for V, $\rho(V,t)$.

$$\rho(V, t+\Delta t \,|\, V_0, t_0) = \underbrace{\int_{-\infty}^{\infty} dV' \rho(V, t+\Delta t \,|\, V', t)\, \rho(V', t \,|\, V_0, t_0)}_{\text{Sum of all the ways to get from } (V_0, t_0) \text{ to } (V, t+\Delta t) \text{ for a Markov process}}$$

How can you go from (V', t) to $(V, t+\Delta t)$ when Δt is very small $(\Delta t \ll \tau_m)$?

We assume it is so small that either there is 1 exc. input, 1 inh. input, or no inputs at all.

So, $$\rho(V, t+\Delta t \,|\, V', t) = P_0 \cdot \delta(V - V_0') + P_+ \cdot \delta(V - V_+') + P_- \, \delta(V - V_-')$$

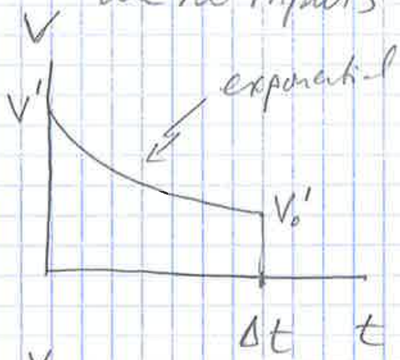
where

$P_0 = Pr(\text{no input}) = 1 - [c_e \nu_e + c_I \nu_I]\Delta t$

$P_+ = Pr(1 \text{ exc. input}) = c_e \nu_e \Delta t$

$P_- = Pr(1 \text{ inh. input}) = c_i \nu_i \Delta t$

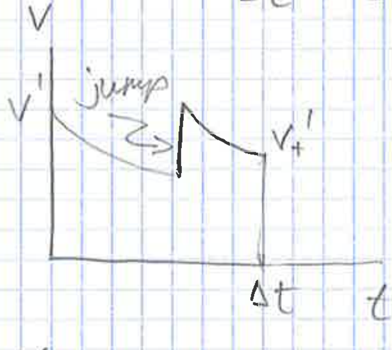
$V_0' \equiv$ voltage after Δt when you start at V' and there are no inputs



Using Eq. 1

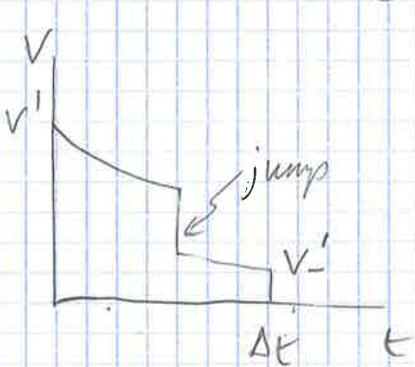
$$V_0' = V' e^{-\Delta t/\tau_m} \approx V'\left(1 - \frac{\Delta t}{\tau_m}\right)$$

small



$$V_+' = V'\left(1 - \frac{\Delta t}{\tau_m}\right) + J_e + o(\Delta t)$$

there's a part here which is the decay of the jump J_e, but we don't need it



$$V_-' = V'\left(1 - \frac{\Delta t}{\tau_m}\right) - J_i + o(\Delta t)$$

then,

$$\int_{-\infty}^{\infty} dV' \rho(V, t+\Delta t | V', t) \rho(V', t | V_0, t_0)$$

I'm leaving off the $|V_0, t_0$ here.

$$= P_0 \int_{-\infty}^{\infty} dV' \delta\left(V - V'\left(1 - \frac{\Delta t}{\tau_m}\right)\right) \rho(V', t) \quad \text{(A)}$$

$$+ P_+ \int_{-\infty}^{\infty} dV' \delta\left(V - V'\left(1 - \frac{\Delta t}{\tau_m}\right) - J_e\right) \rho(V', t) \quad \text{(B)}$$

$$+ P_- \int_{-\infty}^{\infty} dV' \delta\left(V - V'\left(1 - \frac{\Delta t}{\tau_m}\right) + J_i\right) \rho(V', t) \quad \text{(C)}$$

for (A) let $x = V'\left(1 - \frac{\Delta t}{\tau_m}\right)$, $dx = dV'\left(1 - \frac{\Delta t}{\tau_m}\right)$

$V' = x\left(1 + \frac{\Delta t}{\tau_m}\right) + O(\Delta t^2)$ $\Rightarrow dV' = dx\left(1 + \frac{\Delta t}{\tau_m}\right) + O(\Delta t^2)$

for (B) let $x = V'\left(1 - \frac{\Delta t}{\tau_m}\right) + J_e$

for (C) let $x = V'\left(1 - \frac{\Delta t}{\tau_m}\right) - J_i$

$$\Rightarrow \rho(V, t+\Delta t) = P_0\left(1 + \frac{\Delta t}{\tau_m}\right) \int_{-\infty}^{\infty} dx\, \delta(V - x)\, \rho\left(x\left(1 + \frac{\Delta t}{\tau_m}\right), t\right)$$

$$+ P_+\left(1 + \frac{\Delta t}{\tau_m}\right) \int_{-\infty}^{\infty} dx\, \delta(V - x)\, \rho\left(x\left(1 + \frac{\Delta t}{\tau_m}\right) - J_e, t\right)$$

$$+ P_-\left(1 + \frac{\Delta t}{\tau_m}\right) \int_{-\infty}^{\infty} dx\, \delta(V - x)\, \rho\left(x\left(1 + \frac{\Delta t}{\tau_m}\right) + J_i, t\right)$$

We won't need these terms since $P_+, P_- \sim O(\Delta t)$ already and we will only keep terms of $O(\Delta t)$

$$\rho(V,t+\Delta t) = \left(1-[c_e\nu_e + c_i\nu_i]\Delta t\right)\rho\left(V\left(1+\frac{\Delta t}{\tau_m}\right),t\right)\cdot\left(1+\frac{\Delta t}{\tau_m}\right)$$
$$+\, c_e\nu_e\Delta t\,\rho(V-J_e,t) + c_i\nu_i\Delta t\,\rho(V+J_i,t)$$

expand in Taylor Series for small Δt

$$\rho(V,t) + \frac{\partial\rho}{\partial t}(V,t)\Delta t = \left(1-[c_e\nu_e + c_i\nu_i]\Delta t\right)\left(1+\frac{\Delta t}{\tau_m}\right)\left(\rho(V,t) + \frac{\partial\rho}{\partial V}(V,t)\cdot\frac{V\Delta t}{\tau_m} + O(\Delta t^2)\right)$$
$$+\, c_e\nu_e\Delta t\,\rho(V-J_e,t)$$
$$+\, c_i\nu_i\Delta t\,\rho(V+J_i,t)$$

$$\cancel{\rho(V,t)} + \frac{\partial\rho}{\partial t}(V,t)\Delta t = \cancel{\rho(V,t)} + \frac{\partial\rho}{\partial V}(V,t)\cdot\frac{V\Delta t}{\tau_m} + \frac{\Delta t}{\tau_m}\rho(V,t)$$
$$+\, c_e\nu_e\Delta t\left[\rho(V-J_e,t)-\rho(V,t)\right]$$
$$+\, c_i\nu_i\Delta t\left[\rho(V+J_i,t)-\rho(V,t)\right] + O(\Delta t^2)$$

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial V}\left(\frac{V}{\tau_m}\rho\right) + c_e\nu_e\left(\rho(V-J_e,t)-\rho(V,t)\right) + c_i\nu_i\left(\rho(V+J_i,t)-\rho(V,t)\right) + O(\Delta t) \qquad (2)$$

Then take $\Delta t \to 0$

If we assume $J_e \ll 1$ we can Taylor expand $\rho(V-J_e,t)$

$$\rho(V-J_e,t) = \rho(V,t) - J_e\frac{\partial\rho}{\partial V}(V,t) + \frac{J_e^2}{2}\frac{\partial^2\rho}{\partial V^2}(V,t) + O(J_e^3)$$

If we truncate at second order we can write Eq. 2 as

$$\frac{\partial\rho}{\partial t} = \frac{\sigma^2}{2\tau_m}\frac{\partial^2\rho}{\partial V^2} + \frac{\partial}{\partial V}\left(\left(\frac{V-\mu}{\tau_m}\right)\rho\right) \qquad (3)$$

$$\sigma^2 = \tau_m(J_e^2 c_e\nu_e + J_i^2 c_i\nu_i)$$
$$\mu = \tau_m(J_e c_e\nu_e - J_i c_i\nu_i)$$

Eq. 3 is the Fokker-Planck equation. It arises through the diffusion approximation, i.e. $J_e, J_i \ll 1$ but $J_e c_e\nu_e$ and $J_i c_i\nu_i \sim O(1)$

B. Steady-state Solution to Fokker-Plank Equation

$$\frac{\partial \rho}{\partial t} = \frac{\sigma^2}{2\tau}\frac{\partial^2 \rho}{\partial V^2} + \frac{\partial}{\partial V}\left(\left(\frac{V-\mu}{\tau}\right)\rho\right) \qquad (4)$$

can also be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial V} = 0$$

$$F = -\frac{\sigma^2}{2\tau}\frac{\partial \rho}{\partial V} - \frac{(V-\mu)}{\tau}\rho$$

Boundary Conditions:

- $\boxed{\rho(V_t, t) = 0}$ (ρ is zero for all voltages $> V_t$ and so must be zero at V_t, otherwise the gradient $\frac{\partial \rho}{\partial V}$ would be infinite $\Rightarrow$ inf. flux $\rightarrow$ inf. firing rate)

- $\nu(t) = F(V_t, t) = -\frac{\sigma^2}{2\tau}\frac{\partial \rho(V_t,t)}{\partial V} - \frac{(V_t-\mu)}{\tau}\underbrace{\rho(V_t,t)}_{=0}$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial V}(V_t, t) = -\frac{2\tau\nu(t)}{\sigma^2}}$$ (the flux at threshold is just the firing rate)

But the flux at threshold gets reinjected at the reset potential, so there is a jump in the flux there, i.e.

$$\lim_{\epsilon \to 0} \quad F(V_r+\epsilon, t) - F(V_r-\epsilon, t) = \nu(t)$$

$$-\frac{\sigma^2}{2\tau}\frac{\partial \rho}{\partial V}(V_r+\epsilon, t) - \frac{(V_r+\epsilon-\mu)}{\tau}\rho(V_r+\epsilon, t)$$

$$+\frac{\sigma^2}{2\tau}\frac{\partial \rho}{\partial V}(V_r-\epsilon, t) + \frac{(V_r-\epsilon-\mu)}{\tau}\rho(V_r-\epsilon, t)$$

$$\Rightarrow \lim_{\epsilon\to 0} \boxed{\frac{\partial \rho}{\partial V}(V_r+\epsilon,t) - \frac{\partial \rho}{\partial V}(V_r-\epsilon,t) = -\frac{2\tau\nu(t)}{\sigma^2}}$$

- $$\boxed{\int_{-\infty}^{V_t} dV\, \rho(V,t) = 1}$$ (ρ is a probability distribution or prob. density)

if there is a refractory time τ_{ref} then this condition changes to

$$\boxed{\int_{-\infty}^{V_t} dV\, \rho(V,t) + \tau_{ref}\nu(t) = 1}$$

- We also require that $\rho \to 0$ as $V \to \infty$ but this will be fulfilled trivially

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Now we solve for the steady-state, i.e. $\rho(V,t) = \rho_0(V)$, $\nu(t) = \nu_0$

From Eq. 4

$$\frac{\partial \rho_0}{\partial V} + 2\frac{(V-\mu)}{\sigma^2}\rho_0 = C_1$$

What is C_1? Since the flux has a jump at V_r, C_1 takes on one value for $V < V_r$ and another for $V \geq V_r$

$$\frac{\partial \rho_0}{\partial V}(V_t) + 2\frac{(V_t-\mu)}{\sigma^2}\underbrace{\rho_0(V_t)}_{=0} = C_1$$

$$\Rightarrow \boxed{C_1 = -\frac{2\tau\nu_0}{\sigma^2}} \qquad V \geq V_r$$

this implies $C_1 = 0 \quad V < V_r$

So we can write

$$\frac{\partial \rho_0}{\partial V} + \frac{2(V-\mu)}{\sigma^2}\rho_0 = -\frac{2\tau\nu_0}{\sigma^2} H(V-V_r),$$

where $H(x) = \begin{cases} 0 & x<0 \\ 1 & x\geq 0 \end{cases}$

Integrating,

$$\frac{\partial}{\partial V}\left(\rho_0 e^{\frac{(V-\mu)^2}{\sigma^2}}\right) = -\frac{2\tau\nu_0}{\sigma^2} e^{\frac{(V-\mu)^2}{\sigma^2}} H(V-V_r)$$

$$\rho_0 e^{\frac{(V-\mu)^2}{\sigma^2}} = -\frac{2\tau\nu_0}{\sigma^2}\int^{V} dx\, e^{\frac{(x-\mu)^2}{\sigma^2}} H(x-V_r)$$

$$\rho_0(V) = +\frac{2\tau\nu_0}{\sigma^2} e^{-\frac{(V-\mu)^2}{\sigma^2}} \int_{V}^{V_t} dx\, e^{\frac{(x-\mu)^2}{\sigma^2}} H(x-V_r)$$

let $y = \frac{x-\mu}{\sigma}$ $\quad dx = \sigma\, dy$

$$\boxed{\rho_0(V) = \frac{2\tau\nu_0}{\sigma} e^{-\frac{(V-\mu)^2}{\sigma^2}} \int_{\frac{V-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} dy\, e^{y^2} H\left(y - \left(\frac{V_r-\mu}{\sigma}\right)\right)}$$

To find the firing rate we use the condition

$$\int_{-\infty}^{V_t} dV \rho_0(V) = 1$$

$$\frac{2\tau\nu_0}{\sigma}\int_{-\infty}^{V_t} dV e^{-\frac{(V-\mu)^2}{\sigma^2}} \int_{\frac{V-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} dy\, e^{y^2} H\left(y - \left(\frac{V_r-\mu}{\sigma}\right)\right)$$

let $u = \frac{V-\mu}{\sigma}$ $dV = \sigma\, du$

$$\Rightarrow 2\tau\nu_0 \int_{-\infty}^{\frac{V_t-\mu}{\sigma}} du\, e^{-u^2} \int_{u}^{\frac{V_t-\mu}{\sigma}} dy\, e^{y^2} H\left(y - \left(\frac{V_r-\mu}{\sigma}\right)\right)$$

$$-\infty < u \leq \frac{V_t-\mu}{\sigma}$$

$$u < y \leq \frac{V_t-\mu}{\sigma}$$

Inverting order of integration

$$2\tau\nu_0 \int_{-\infty}^{\frac{V_t-\mu}{\sigma}} dy\, e^{y^2} H\left(y - \left(\frac{V_r-\mu}{\sigma}\right)\right) \int_{-\infty}^{y} du\, e^{-u^2}$$

check this with Wikipedia!

$$= \cancel{2}\tau\nu_0 \int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} dy\, e^{y^2} \cdot \frac{\sqrt{\pi}}{\cancel{2}} \operatorname{erfc}(-y)$$

$$\Rightarrow \tau\nu_0\sqrt{\pi} \int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} dy\, e^{y^2} \operatorname{erfc}(-y) = 1$$

$$\nu_0 = \frac{1}{\tau\sqrt{\pi}} \left[\int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} dy\, e^{y^2} \operatorname{erfc}(-y) \right]^{-1} \quad (5)$$

C. Here we calculate the CV of the ISI using the recurrence relation Eq. 15.27 from Renart et al

First note that I will call $\langle T_{fp}^{k} \rangle$ (kth moment of mean first passage time) just μ_K

So

$$\frac{\sigma^2}{2}\frac{d^2\mu_1}{dx^2} - (x-\mu)\frac{d\mu_1}{dx} = -1$$ (eq. 15.27 in Renart et al.)

let $z = \frac{d\mu_1}{dx}$

$$\frac{dz}{dx} - \frac{2(x-\mu)}{\sigma^2}z = -\frac{2}{\sigma^2}$$

$$\frac{d}{dx}\left(z e^{-\frac{(x-\mu)^2}{\sigma^2}}\right) = -\frac{2}{\sigma^2}e^{-\frac{(x-\mu)^2}{\sigma^2}}$$

$$z e^{-\frac{(x-\mu)^2}{\sigma^2}} = -\frac{2}{\sigma^2}\int_{-\infty}^{x} dy\, e^{-\frac{(y-\mu)^2}{\sigma^2}}$$

$$z = -\frac{2}{\sigma^2} e^{\frac{(x-\mu)^2}{\sigma^2}}\int_{-\infty}^{x} dy\, e^{-\frac{(y-\mu)^2}{\sigma^2}}$$

$$\mu_1 = -\frac{2}{\sigma^2}\int_{V_t}^{x} du\, e^{\frac{(u-\mu)^2}{\sigma^2}}\int_{-\infty}^{u} dy\, e^{-\frac{(y-\mu)^2}{\sigma^2}}$$

$$= \frac{2}{\sigma^2}\int_{V_r}^{V_t} du\, e^{\frac{(u-\mu)^2}{\sigma^2}}\int_{-\infty}^{u} dy\, e^{-\frac{(y-\mu)^2}{\sigma^2}}$$

x is V_r

let $w = \frac{y-\mu}{\sigma}$

$v = \frac{u-\mu}{\sigma}$

$$= 2\int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} dv\, e^{v^2}\underbrace{\int_{-\infty}^{v} dw\, e^{-w^2}}_{\frac{\sqrt{\pi}}{2}\mathrm{erfc}(-v)} = \sqrt{\pi}\int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} dv\, e^{v^2}\,\mathrm{erfc}(-v)$$

So we have $\mu_1 = \dfrac{1}{\nu_0 \tau}$

Now

$$\frac{\sigma^2}{2}\frac{d^2\mu_2}{dx^2} - (x-\mu)\frac{d\mu_2}{dx} = -2\mu_1(x)$$

let $z = \dfrac{d\mu_2}{dx}$

$$\frac{dz}{dx} - \frac{2}{\sigma^2}(x-\mu)z = -\frac{4}{\sigma^2}\mu_1(x)$$

$$\frac{d}{dx}\left(z e^{-\frac{(x-\mu)^2}{\sigma^2}}\right) = -\frac{4}{\sigma^2} e^{-\frac{(x-\mu)^2}{\sigma^2}} \mu_1(x)$$

$$z e^{-\frac{(x-\mu)^2}{\sigma^2}} = -\frac{4}{\sigma^2}\int_{-\infty}^{x} dy\, e^{-\frac{(y-\mu)^2}{\sigma^2}} \mu_1(y)$$

$$z = -\frac{4}{\sigma^2} e^{\frac{(x-\mu)^2}{\sigma^2}} \int_{-\infty}^{x} dy\, e^{-\frac{(y-\mu)^2}{\sigma^2}} \mu_1(y)$$

$$\mu_2 = -\frac{4}{\sigma^2}\int_{V_t}^{x} dx'\, e^{\frac{(x'-\mu)^2}{\sigma^2}} \int_{-\infty}^{x'} dy\, e^{-\frac{(y-\mu)^2}{\sigma^2}} \mu_1(y)$$

$$= \frac{4}{\sigma^2}\int_{x}^{V_t} dx'\, e^{\frac{(x'-\mu)^2}{\sigma^2}} \int_{-\infty}^{x'} dy\, e^{-\frac{(y-\mu)^2}{\sigma^2}} \cdot \sqrt{\pi}\int_{\frac{y-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} dv\, e^{v^2} \operatorname{erfc}(-v)$$

let $w = \dfrac{y-\mu}{\sigma}$

$u = \dfrac{x'-\mu}{\sigma}$

$$\mu_2 = 4\sqrt{\pi}\int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} du\, e^{u^2} \int_{-\infty}^{u} dw\, e^{-w^2} \int_{w}^{\frac{V_t-\mu}{\sigma}} dv\, e^{v^2} \operatorname{erfc}(-v)$$

Note $\int_{W}^{\frac{V_t-\mu}{\sigma}} (\;) = \int_{W}^{\frac{V_r-\mu}{\sigma}} (\;) + \int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} (\;)$

$$\Rightarrow \mu_2 = 4\sqrt{\pi} \int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} du\, e^{u^2} \int_{-\infty}^{u} dw\, e^{-w^2} \int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} dv\, e^{v^2} \operatorname{erfc}(-v)$$

$\frac{\sqrt{\pi}}{2} \operatorname{erfc}(-u)$

$$+ 4\sqrt{\pi} \int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} du\, e^{u^2} \int_{-\infty}^{u} dw\, e^{-w^2} \int_{W}^{\frac{V_r-\mu}{\sigma}} dv\, e^{v^2} \operatorname{erfc}(-v)$$

$-\infty < V \leq u$

$-\infty < W < V \leq u$

$$= 2\pi \left(\int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} du\, e^{u^2} \operatorname{erf}(-u) \right)^2 + 4\sqrt{\pi} \int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} du\, e^{u^2} \int_{-\infty}^{u} dv\, e^{v^2} \operatorname{erfc}(-v) \int_{-\infty}^{v} dw\, e^{-w^2}$$

$\frac{\sqrt{\pi}}{2} \operatorname{erfc}(-v)$

factor 2 should not be here

$$= 2\mu_1^2 + 2\pi \int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} du\, e^{u^2} \int_{-\infty}^{u} dv\, e^{v^2} [\operatorname{erfc}(-v)]^2$$

$$CV = \frac{\mu_2 - \mu_1^2}{\mu_1^2} = \sqrt{2\pi \tau^2 \nu_0^2 \int_{\frac{V_r-\mu}{\sigma}}^{\frac{V_t-\mu}{\sigma}} du\, e^{u^2} \int_{-\infty}^{u} dv\, e^{v^2} [\operatorname{erfc}(-v)]^2}$$

(6)

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D. All-to-all network of LIF neurons

$$\tau_E \frac{dV_i^E}{dt} = -V_i^E + \mu_{ext,E} + \tau\left(\frac{J_e}{N}\sum_{j=1}^{N_e}\sum_k \delta(t-t_j^k-D) - \frac{J_i}{N}\sum_{j=1}^{N_i}\sum_k \delta(t-t_j^k-D)\right) + \sigma\xi_i(t)$$

and an analogous equation for inh. neurons

For the sims I chose $\tau_E = \tau_I = 20\,ms$

$V_t = 20\,mV$, $V_r = 10\,mV$

$J_e = J_i = 30$, $I_{ext,I} = 14{,}7\,mV$

$\sigma = 5\,mV$, $\xi_i(t)$ is a Gaussian white noise process with zero mean and unit variance. So there is no diffusion approximation here.

E. Sparse Network of LIF neurons

The difference with the all-to-all network is twofold.

1- Recurrent inputs

The recurrent input to an excitatory neuron i is

$$I_i^E = \tau J_e \sum_{j=1}^{N_e} W_{ij}^{EE} \sum_k \delta(t-t_j^k-D) - \tau J_i \sum_{j=1}^{N_i} W_{ij}^{EI} \sum_k \delta(t-t_j^k-D)$$

$W_{ij}^{EE} = 1$ with probability $p_{EE} = \frac{C_{EE}}{N_E}$ otherwise it is zero

2- External inputs

I take C_{ext} independent Poisson processes with rate ν_{ext}^E and amplitude J_{ext} for E neurons. For I neurons it is the same but with ν_{ext}^I.

For the theory, now

$$\mu_E = \tau\left(J_{ext} C_{ext} \nu_{ext} + J_{ee} C_{ee} \nu_e - J_{ei} C_{ei} \nu_i\right)$$

$$\sigma_E^2 = \tau\left(J_{ext}^2 C_{ext} \nu_{ext} + J_{ee}^2 C_{ee} \nu_e + J_{ei}^2 C_{ei} \nu_i\right)$$

with analogous equations for I neurons.
For the simulation I take $J_{ee} = J_{ie} = J_e$

$J_{ii} = J_{ei} = J_i$

F. Linear Stability for All-to-all network

$$\frac{\partial p_e}{\partial t} = \frac{\sigma^2}{2\tau}\frac{\partial^2 p_e}{\partial V^2} + \frac{\partial}{\partial V}\left[\frac{(V-\mu_e)}{\tau}p_e\right] \quad (7)$$

$$p_e(V_t, t) = 0$$

$$\frac{\partial p_e}{\partial V}(V_t, t) = -\frac{2\tau\nu_e(t)}{\sigma^2}$$

$$\frac{\partial p_e}{\partial V}(V_r^+, t) - \frac{\partial p_e}{\partial V}(V_r^-, t) = -\frac{2\tau\nu_e(t)}{\sigma^2}$$

$$\int_{-\infty}^{V_t} dV\, p_e(V, t) = 1$$

with an analogous equation and set of B.C.'s for the inhibitory population.

Now

$$p_e(V,t) = p_0(V) + \epsilon\, p_1(V) e^{\lambda t}$$

$$\nu_e(t) = \nu_{e0} + \epsilon\, \nu_{e1} e^{\lambda t}$$

$$\mu_e = \mu_{e0} + \epsilon\, \mu_{e1} e^{\lambda t}$$

$$\mu_e = \mu_{ext} + \tau(J_e\nu_e - J_i\nu_i)$$

$$= \mu_{ext} + \tau(J_e\nu_{e0} - J_i\nu_{i0}) + \epsilon(J_e\nu_{e1} - J_i\nu_{i1})e^{\lambda t}$$

$$\Rightarrow \mu_{e1} = J_e\nu_{e1} - J_i\nu_{i1}$$

Plug into Eqs. 7 and collect terms of order ϵ
(we've already calculated $O(1)$ terms!)

$$\lambda \rho_1(V) = \frac{\sigma^2}{2\tau}\frac{\partial^2 \rho_1}{\partial V^2} + \frac{\partial}{\partial V}\left[\left(\frac{V-\mu_{e0}}{\tau}\right)\rho_1\right] - \frac{\mu_{e1}}{\tau}\frac{\partial \rho_0}{\partial V} \quad (8)$$

$$\rho_1(V_t) = 0$$

$$\frac{\partial \rho_1}{\partial V}(V_t) = -\frac{2\tau \nu_{e1}}{\sigma^2}$$

$$\frac{\partial \rho_1}{\partial V}(V_r^+) - \frac{\partial \rho_1}{\partial V}(V_r^-) = -\frac{2\tau \nu_{e1}}{\sigma^2}$$

$$\int_{-\infty}^{V_t} dV \rho_1(V) = 0$$

You must solve Eqs. 8 to find $\rho_1(V)$, ν_{e1}, ν_1, etc. The solutions can be expressed in terms of Confluent Hypergeometric functions or in integral form if solved via Laplace transforms. See Brunel 2000 for more details. However all of this is a lot of work. It's much easier to solve (8) numerically!

G. Numerical Solution of Fokker-Planck Eq.

(Following Richardson PRE 2007)

First, we rewrite Eq. 7 as

$$\frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial V} = \nu(t)\left(\delta(V-V_r) - \delta(V-V_t)\right)$$

$$F = -\frac{\sigma^2}{2\tau}\frac{\partial \rho}{\partial V} - \frac{(V-\mu)}{\tau}\rho$$

This takes care of jump conditions at V_t and V_r

We want to find the stationary state and its stability numerically.

$$\rho = \rho_0 + \epsilon \rho_1 e^{i\omega t}$$

$$\nu = \nu_0 + \epsilon \nu_1 e^{i\omega t}$$

$$\mu = \mu_0 + \epsilon \mu_1 e^{i\omega t}$$

$$F = F_0 + \epsilon F_1 e^{i\omega t}$$

I assume $\lambda = \lambda_r + i\omega$, with $\lambda_r = 0$, so I'm right at the instability