

Thermodynamic limits of networks of rate (Hopfield, Wilson-Cowan) neurons I: independent weights

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Isolated rate neuron

The dynamics

$$\mathcal{S} := \begin{cases} dV_t & = -\alpha V_t dt + \sigma dB_t, \quad 0 \leq t \leq T \\ \text{Law of } V_0 & = \mu_0, \end{cases}$$

- ▶ B_t is a standard Wiener/Brownian process
- ▶ σ is the intensity of the noise
- ▶ α controls how fast the neuron relaxes to the initial condition

The solution of \mathcal{S} is

$$V_t = \exp(-\alpha t) V_0 + \sigma \int_0^t \exp(\alpha(s-t)) dB_s.$$

Coupling N neurons

$$S^N(J, \chi) := \begin{cases} dV_t^i & = -\alpha V_t^i dt + \chi_i dt + \sigma dB_t^i + \sum_{j=1}^N J_{ij} f(V_t^j) dt \\ \text{Law of } V_N(0) & = \mu_0^{\otimes N} \end{cases}$$

- ▶ f is a sigmoid: Lipschitz continuous.
- ▶ The χ_i s are i.i.d. $\mathcal{N}(\bar{\chi}, \sigma_\chi^2)$ Gaussian random variables: injected currents.

Coupling N neurons

- ▶ The J_{ij} s are i.i.d $\mathcal{N}(\frac{\bar{J}}{N}, \frac{J^2}{N})$ Gaussian random variables: synaptic weights
- ▶ The B_t^i s are independent standard Wiener/Brownian processes
- ▶ We note $(\Omega, \mathcal{A}, \gamma)$ the underlying probability space, e.g.

$$\bar{\chi} = \int_{\Omega} \chi^i(\omega) d\gamma(\omega)$$

Well-posedness of the finite size problem

- ▶ The function f is Lipschitz continuous, hence
- ▶ so is in \mathbb{R}^N the drift term of $\mathcal{S}^N(J, \chi)$, hence
- ▶ there is a unique solution to this set of N coupled stochastic differential equations (SDEs).
- ▶ The solution $V_N(t) = (V_t^1, \dots, V_t^N)$ is continuous on $[0, T]$. We note \mathcal{T} the set $\mathcal{C}([0, T]; \mathbb{R})$.

Well-posedness of the finite size problem

- ▶ We note $P^N(J, \chi)$ the law of this solution on $[0, T]$.
- ▶ It is a (random) element of $\mathcal{P}(\mathcal{T}^N)$ where \mathcal{T}^N is the set of elements $u_N = (u^1, \dots, u^N)$ where each u^i , $i = 1, \dots, N$ is an element of \mathcal{T} .
- ▶ The law of the solution to \mathcal{S} is an element of $\mathcal{P}(\mathcal{T})$ noted P .

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Empirical measure

Define the empirical measure:

$$\hat{\mu}_N(V_N) = \frac{1}{N} \sum_{i=1}^N \delta_{V^i}$$

It is a probability measure on \mathcal{T}

Example:

$$\int_{\mathcal{T}} v_t v_s \hat{\mu}_N(V_N)(dv) = \frac{1}{N} \sum_{i=1}^N V_t^i V_s^i$$

Goal

To study the law of $\hat{\mu}_N(V_N)$ under $P^N(J, \chi)$

Consider $\Pi^N(J, \chi)$ the probability on $\mathcal{P}(\mathcal{T})$ defined by

$$\Pi^N(J, \chi)(B) = P^N(J, \chi)(\hat{\mu}_N(V_N) \in B) \quad \forall B \in \mathcal{B}(\mathcal{P}(\mathcal{T}))$$

This is too complicated.

Goal

We first study the law of $\hat{\mu}_N(V_N)$ under Q^N the annealed law of $P^N(J, \chi)$ w.r.t. the synaptic weights J and the injected currents χ . Consider Π^N the probability on $\mathcal{P}(\mathcal{T})$ defined by

$$\Pi^N(B) = Q^N(\hat{\mu}_N(V_N) \in B) = \int_{\Omega} P^N(J(\omega), \chi(\omega))(\hat{\mu}_N(V_N) \in B) d\gamma(\omega)$$

The Cameron-Martin-Girsanov theorem

Theorem

Assume $B_t = (B_t^i)_{i=1\dots N}$ is an N -dimensional Brownian defined on $(\Omega, \mathcal{A}, \mathcal{A}_t, P)$ and let $\Phi = (\Phi^1, \dots, \Phi^N)$ be in $\mathbb{L}^2([0, T], \mathbb{R}^N)$.

Define

$$\zeta_0^T = \int_0^T \langle \Phi(t), dB_t \rangle - \frac{1}{2} \int_0^T |\Phi(t)|^2 dt$$

$$\tilde{B}_t = B_t - \int_0^t \Phi(u) du$$

$$d\tilde{P}(\omega) = \exp(\zeta_0^T) dP(\omega)$$

If $\tilde{P}(\Omega) = 1$ then \tilde{B}_t is a N -dimensional Brownian motion on $(\Omega, \mathcal{A}, \tilde{P})$.

The Cameron-Martin-Girsanov theorem

- ▶ $\exp(\zeta_0^T)$ is the Radon-Nikodym derivative of \tilde{P} with respect to P .
- ▶ \tilde{P} is said to be absolutely continuous with respect to P

$$\tilde{P} \ll P$$

The Cameron-Martin-Girsanov theorem

A sufficient condition for $\tilde{P}(\Omega) = 1$ to hold is that there exists two positive constants μ and C such that

$$\mathbb{E} \left[e^{\mu \|\Phi(t)\|^2} \right] \leq C \quad \forall t \in [0, T]$$

The Girsanov theorem

Theorem

Let $B_t = (B_t^i)_{i=1\dots N}$ be an N -dimensional Brownian defined on $(\Omega, \mathcal{A}, \mathcal{A}_t, P)$. Let $x(t)$ be the N -dimensional Itô process given by

$$x(t) = x(0) + \int_0^t g(s) ds + \int_0^t h(s) dB(s)$$

with g in $\mathbb{L}^2([0, T], \mathbb{R}^N)$, and h in $\mathbb{L}^2([0, T], \mathbb{R}^{N \times N})$. Let $\Phi \in \mathbb{L}^2([0, T], \mathbb{R}^N)$. Let \tilde{B}_t and \tilde{P} be defined as before. If $\tilde{P}(\Omega) = 1$ then $x(t)$ is still an Itô process on $(\Omega, \mathcal{A}, \mathcal{A}_t, \tilde{P})$. More precisely

$$x(t) = x(0) + \int_0^t (g(s) + h(s)\Phi(s)) ds + \int_0^t h(s) d\tilde{B}(s)$$

Annealed law

- ▶ Apply Girsanov theorem to $\mathcal{S}^N(J, \chi)$ in order to obtain N times the uncoupled system \mathcal{S} .
- ▶ It follows that $P^N(J, \chi)$ is absolutely continuous w.r.t. $P^{\otimes N}$ and

$$\frac{dP^N(J, \chi)}{dP^{\otimes N}} = \exp \sum_{i=1}^N \left\{ \frac{1}{\sigma} \int_0^T \left(\chi_i + \sum_{j=1}^N J_{ij} f(V_t^j) \right) dB_t^i - \frac{1}{2\sigma^2} \int_0^T \left(\chi_i + \sum_{j=1}^N J_{ij} f(V_t^j) \right)^2 dt \right\}$$

- ▶ The right hand side being a measurable function of (J, χ) , $Q^N = \int_{\Omega} P^N(J(\omega), \chi(\omega)) d\omega$ is well-defined.

Topologies

We endow \mathcal{T} with the topology of uniform convergence:

$$\|u\| = \sup_{t \in [0, T]} |u_t|, \quad d(u, v) = \|u - v\|$$

and $\mathcal{P}(\mathcal{T})$ with the Wasserstein-1 distance

$$D(\mu, \nu) = \inf_{\xi} \int_{\mathcal{T} \times \mathcal{T}} d(u, v) d\xi(u, v),$$

where ξ is a coupling between μ and ν .

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Fundamental relation: I

For fixed values of $(V_{N,t})$ consider the N random processes

$$G_t^{N,i} = \chi_i + \sum_{j=1}^N J_{ij} f(V_t^j)$$

They are i.i.d. Gaussian processes with mean

$$\mathbb{E} \left[G_t^{N,i} \right] = \bar{\chi} + \frac{\bar{J}}{N} \sum_{j=1}^N f(V_t^j) = \bar{\chi} + \bar{J} \int_{\mathcal{T}} f(v_t) \hat{\mu}_N(V_N)(dv),$$

and covariance

$$\begin{aligned} \text{cov}(G_t^{N,i} G_s^{N,i}) &= \sigma_\chi^2 + \frac{J^2}{N} \sum_{j=1}^N f(V_t^j) f(V_s^j) = \\ &\sigma_\chi^2 + J^2 \int_{\mathcal{T}} f(v_t) f(v_s) \hat{\mu}_N(V_N)(dv) \end{aligned}$$

Fundamental relation: II

Notation: $\gamma_{\hat{\mu}_N(V_N)}$ is the probability law on (Ω, \mathcal{A}) under which the Gaussian processes $G_t^{N,i}$ are i.i.d with the previous mean and covariance.

Fundamental relation: II

Lemma

$$\frac{dQ^N}{dP^{\otimes N}}(V_N) = \exp N\Gamma(\hat{\mu}_N(V_N)),$$

where

$$\Gamma(\hat{\mu}_N(V_N)) =$$

$$\int \log \int \exp \left(\frac{1}{\sigma} \int_0^T G_t^N(\omega) dB_t(v) - \frac{1}{2\sigma^2} \int_0^T G_t^N(\omega)^2 dt \right) d\gamma_{\hat{\mu}_N}(\omega) d\hat{\mu}_N(v),$$

and

$$\sigma dB_t(v) = dv_t + \alpha v_t dt$$

Fundamental relation: III

Proof:

By Girsanov theorem

$$\frac{dQ^N}{dP^{\otimes N}} = \int \exp \sum_{i=1}^N \left\{ \frac{1}{\sigma} \int_0^T G_t^{N,i}(\omega) dB_t(V^i) - \frac{1}{2\sigma^2} \int_0^T G_t^{N,i}(\omega)^2 dt \right\} d\gamma_{\hat{\mu}_N(V_N)}(\omega)$$

i.i.d. of the $(G_t^{N,i})$:

$$\prod_{i=1}^N \int \exp \left\{ \frac{1}{\sigma} \int_0^T G_t^N(\omega) dB_t(V^i) - \frac{1}{2\sigma^2} \int_0^T G_t^N(\omega)^2 dt \right\} d\gamma_{\hat{\mu}_N(V_N)}(\omega)$$

Fundamental relation: IV

Take the log:

$$\sum_{i=1}^N \log \int \exp \left\{ \frac{1}{\sigma} \int_0^T G_t^N(\omega) dB_t(V^i) - \frac{1}{2\sigma^2} \int_0^T G_t^N(\omega)^2 dt \right\} d\gamma_{\hat{\mu}_N}(\omega) =$$

$$N \int \log \int \exp \left\{ \frac{1}{\sigma} \int_0^T G_t^N(\omega) dB_t(v) - \frac{1}{2\sigma^2} \int_0^T G_t^N(\omega)^2 dt \right\} d\gamma_{\hat{\mu}_N}(\omega) \hat{\mu}_N(dv)$$

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Definition of Γ : I

Given $\mu \in \mathcal{P}(\mathcal{T})$ define

Mean

$$c_\mu(t) = \bar{\chi} + \bar{J} \int f(v_t) d\mu(v)$$

Covariance

$$K_\mu(t, s) = \sigma_\chi^2 + J^2 \int f(v_t) f(v_s) d\mu(v)$$

γ_μ is the probability law on (Ω, \mathcal{A}) such that the process G_t has mean c_μ and covariance K_μ

Definition of Γ : II

Define $\Gamma(\mu)$ by

$$\Gamma(\mu) = \int \log \int \exp \left(\frac{1}{\sigma} \int_0^T G_t(\omega) dB_t(v) - \frac{1}{2\sigma^2} \int_0^T G_t(\omega)^2 dt \right) d\gamma_\mu(\omega) d\mu(v)$$

When $\mu \ll P$, B_t is a semi-martingale under μ and the stochastic integral is well-defined.

Definition of the rate function

Mutual entropy:

$$I(\mu|P) = \begin{cases} \int_{\mathcal{T}} \log \frac{d\mu}{dP} d\mu & \text{if } \mu \ll P \\ \infty & \text{otherwise} \end{cases}$$

Rate function:

$$H(\mu) = \begin{cases} I(\mu|P) - \Gamma(\mu) & \text{if } \mu \ll P \\ \infty & \text{otherwise} \end{cases}$$

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Properties of Γ : I

Lemma

$$\Gamma(\mu) \leq I(\mu|P)$$

Hence $H \geq 0$

Properties of Γ : II

Lemma

$$\Gamma = \Gamma_1 + \Gamma_2$$

with

$$\Gamma_1(\mu) = \log \int \exp \left\{ -\frac{1}{2\sigma^2} \int_0^T G_t(\omega)^2 dt \right\} d\gamma_\mu(\omega) = -\frac{1}{2} \det(\sigma \text{Id} + \bar{K}_\mu),$$

and

$$\Gamma_2(\mu) = \frac{1}{2} \left(\langle \bar{L}_\mu^T c_\mu, c_\mu \rangle_{L^2([0, T])} - \|c_\mu\|_{L^2([0, T])}^2 \right) + \int_0^T L_\mu^t(t, t) dt - \int_0^T \int_0^t (L_\mu^t(t, s))^2 ds dt$$

Properties of Γ : III

Proposition

Γ_1 and Γ_2 are bounded and Lipschitz continuous for the Wasserstein-1 distance.

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Large deviation principle: I

Proposition

For all open sets \mathcal{O} of $\mathcal{P}(\mathcal{T})$

$$- \inf_{\mu \in \mathcal{O}} H(\mu) \leq \liminf_{N \rightarrow \infty} \frac{1}{N} \log \Pi^N(\mathcal{O})$$

Large deviation principle: II

Proposition (1)

The sequence Π^N is exponentially tight.

Large deviation principle: III

Proposition (2)

For every compact set F of $\mathcal{P}(\mathcal{T})$

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \Pi^N(F) \leq - \lim_{\mu \in F} H(\mu)$$

A tightness result

Proposition

For all $\varepsilon > 0$ there exists a compact K_ε of $\mathcal{P}(\mathcal{T})$ such that

$$Q^N(\hat{\mu}_N \in K_\varepsilon^c) \leq \varepsilon$$

for all integers N .

Proof.

For any bounded continuous f on \mathcal{T}^N

$$\int f dQ^N \leq I(Q^N | P^{\otimes N}) + \log \int \exp f dP^{\otimes N}$$

Choose $f = \log(1 + P^{\otimes N}(A)^{-1})\mathbf{1}_A$:

$$Q^N(A) \leq \frac{\log 2 + I(Q^N | P^{\otimes N})}{\log(1 + P^{\otimes N}(A)^{-1})}$$

A tightness result

Because the law of the empirical measure under $P^{\otimes N}$ is exponentially tight we can find a compact K_ε such that

$$P^{\otimes N}(K_\varepsilon^c) \leq \exp -\frac{N}{\varepsilon}$$

$$I(Q^N | P^{\otimes N}) = \int \log \frac{dQ^N}{dP^{\otimes N}} dQ^N = N \int \Gamma(\hat{\mu}_N) dQ^N \leq CN,$$

since Γ is bounded. Hence

$$Q^N(K_\varepsilon^c) \leq \frac{\log 2 + NC}{\log(1 + \exp \frac{N}{\varepsilon})}$$

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Uniqueness and characterization of the minimum of H

Theorem

H achieves its minimum ($= 0$) at the unique point of $\mathcal{P}(\mathcal{T})$ given by

$$\left\{ Q \in \mathcal{P}(\mathcal{T}) \mid Q \ll P, \frac{dQ}{dP} = \int \exp \left\{ \frac{1}{\sigma} \int_0^T G_s dB_s - \frac{1}{2\sigma^2} \int_0^T G_s^2 ds \right\} d\gamma_Q \right\}$$

Characterization of the minimum of H as the solution of an SDE

In the case $c_\mu(t) = 0$, i.e. $\bar{J} = \bar{\chi} = 0$.

Theorem

Q is the solution to the non-Markovian stochastic system defined on $[0, T]$ by

$$(S) \begin{cases} V_t & = V_0 - \alpha \int_0^t V_s ds + \sigma B_t \\ \sigma B_t & = W_t + \int_0^t \int_0^s L_Q^s(s, u) dB_u ds \\ \text{Law of } V & = Q, Q|_{\mathcal{A}_0} = \mu_0 \end{cases}$$

or

$$dV_t = -\alpha V_t dt + \sigma dB_t$$

$$\sigma dB_t = \int_0^t L_Q^t(t, s) dB_s dt + dW_t,$$

where W_t is a Q -Brownian motion.

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Convergence of Π^N

Theorem

The law of the empirical measure $\hat{\mu}_N$ under Q^N converges to δ_Q .

Proof.

For all $\delta > 0$ prove, using propositions (1) and (2), that
 $\lim_{N \rightarrow \infty} Q^N(\hat{\mu}_N \in B(Q, \delta)^c) = 0.$



Annealed propagation of chaos

Definition

Q^N is said to be Q -chaotic if for all $m \geq 2$ and $f_i, i = 1, \dots, m$ in $C_b(\mathcal{T})$

$$\lim_{N \rightarrow \infty} \int_{\mathcal{T}^N} f_1(v^1) \cdots f_m(v^m) dQ^N(v^1, \dots, v^N) = \prod_{i=1}^m \int_{\mathcal{T}} f_i(v) dQ(v)$$

Theorem

Q^N is Q -chaotic.

Proof.

This follows from lemma 3.1 in [Szn84]. □

Note that this has nothing to do with the notion of chaotic solutions to a system of ODEs.

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Extension to replicas

Let r be an integer and consider $Q^{r,N}$ the annealed law of the replicated neuronal dynamics

$$Q^{r,N} = \int_{\Omega} P^N(J(\omega), \chi(\omega))^{\otimes r} d\omega$$

It is a probability law on $(\mathcal{T}^r)^N$.

Definition of the empirical measure

$$\hat{\mu}_N^r : (\mathcal{T}^r)^N \rightarrow \mathcal{P}(\mathcal{T}^r)$$

$$(V_1^i, \dots, V_r^i)_{1 \leq i \leq N} \rightarrow \frac{1}{N} \sum_{i=1}^N \delta_{V_1^i \dots V_r^i}$$

Definition of the limit law

Define Q_r to be

$$Q_r \lll P^{\otimes r} \frac{dQ_r}{dP^{\otimes r}} = \int \exp \left\{ \frac{1}{\sigma} \int_0^T \langle G_t, dB_t \rangle - \frac{1}{2\sigma^2} \int_0^T \|G_t\|^2 dt \right\} d\gamma_{Q_r},$$

where, under γ_{Q_r} is an r -dimensional Gaussian process with mean

$$\mathbf{c}_{Q_r}(t)^i = \bar{\chi} + \bar{J} \int f(v_t^i) dQ_r(v^1, \dots, v^r),$$

and covariance

$$\text{cov}(G_t^i G_s^j) = \int f(v_t^i) f(v_s^j) dQ_r(v^1, \dots, v^r)$$

Convergence of the empirical measure

Theorem

For any integer r , the law of the empirical measure $\hat{\mu}_N^r$ under $Q^{r,N}$ converges to δ_{Q_r} .

Propagation of chaos

As a consequence

Theorem

$Q^{r,N}$ is Q_r -chaotic.

For any bounded continuous functions (F_1, \dots, F_m) on \mathcal{T}^r

$$\lim_{N \rightarrow \infty} \int F_1(v_1^1, \dots, v_r^1) \cdots F_m(v_1^m, \dots, v_r^m) dQ^{r,N}(\mathbf{v}^1, \dots, \mathbf{v}^N) = \prod_{i=1}^m \int F_i(v^1, \dots, v^r) dQ_r(v^1, \dots, v^r)$$

Propagation of chaos

In particular, if $F_i(v^1, \dots, v^r) = \prod_{k=1}^r f_i(v^k)$ for m continuous bounded functions on \mathcal{T}

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\left(\int f_1(v^1) \cdots f_m(v^m) dP^N(J, \chi) \right)^r \right] = \prod_{i=1}^m \int f_i(v^1) \cdots f_i(v^r) dQ_r$$

Quenched results

Rewrite the equation for the dynamics of a single neuron k

$$dV_t^k = -\alpha V_t^k + \sigma dB_t^k + \sum_{i=1}^N J_{ki} \mathbb{E}_{P^N(J, \chi)}[f(V_t^i)] + \sum_{i=1}^N J_{ki} (f(V_t^i) - \mathbb{E}_{P^N(J, \chi)}[f(V_t^i)])$$

Assume the (J_{ki}) are independent of the mean $\mathbb{E}_{P^N(J, \chi)}[f(V_t^i)]$.

$\sum_{i=1}^N J_{ki} \mathbb{E}_{P^N(J, \chi)}[f(V_t^i)]$ is a Gaussian process with mean

$$\frac{\bar{J}}{N} \sum_{i=1}^N \mathbb{E}_{P^N(J, \chi)}[f(V_t^i)] = \bar{J} \int \langle f(v_t), \hat{\mu}_N(V_N) \rangle dP^N(J, \chi)(V_N)$$

which converges almost surely to $\mathbb{E}_Q[f(v_t)]$. We call H the corresponding Gaussian process.

Quenched results

The covariance is given by

$$\frac{J^2}{N} \sum_{i=1}^N \mathbb{E}_{P^N(J, \chi)}[f(V_t^i)] \mathbb{E}_{P^N(J, \chi)}[f(V_s^i)] =$$
$$J^2 \int \langle f(v_t^1) f(v_s^2), \hat{\mu}_N^2 \rangle dP^N(J, \chi)^{\otimes 2}$$

which converges almost surely toward $\mathbb{E}_{Q_2}[f(v_t^1) f(v_s^2)]$

Quenched results

Similarly one shows that $\sum_{i=1}^N J_{ki}(f(V_t^i) - \mathbb{E}_{P^N(J, \chi)}[f(V_t^i)])$ converges to a centered Gaussian process G with covariance $\frac{1}{2} \mathbb{E}_{Q_2}[(f(v_t^1) - f(v_t^2))(f(v_s^1) - f(v_s^2))]$

External input

For $g \in \mathbb{L}^2([0, T])$ note $P(g)$ the law of the solution to the SDE

$$dV_t = -\alpha V_t dt + \sigma dB_t + g(t)dt$$

Note \mathbb{E}^g the expectation over g and define

$$P_H = \mathbb{E}^G[P(G + H)],$$

Relation Q_r and P_H

Theorem

For any integer r

$$Q_r = \mathbb{E}^H [P_H^{\otimes r}]$$

Quenched result

Corollary

For any integer r , for any continuous bounded functions f_1, \dots, f_m on \mathcal{T}

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\left(\int f_1(v^1) \cdots f_m(v^m) dP^N(J, \chi)(v) \right)^r \right] = \prod_{i=1}^m \mathbb{E}^H \left[\left(\int f_i(v) dP_H(v) \right)^r \right]$$

Quenched result

Since the random variables $\int f_1(v^1) \cdots f_m(v^m) dP^N(J, \chi)(v)$ are bounded, this is equivalent to the convergence in law of these random variables.

Theorem

For any continuous bounded functions f_1, \dots, f_m on \mathcal{T} , $\int f_1(v^1) \cdots f_m(v^m) dP^N(J, \chi)(v)$ converges in law, when N goes to infinity, to $\prod_{i=1}^m \int f_i(v) dP_{H_i}(v)$, where H_i are independent copies of the Gaussian process described above.

This means that for all $\varepsilon > 0$

$$\lim_{N \rightarrow \infty} \gamma \left(\omega \mid \left| \int f_1(v^1) \cdots f_m(v^m) dP^N(J(\omega), \chi(\omega))(v) - \prod_{i=1}^m \int f_i(v) dP_{H_i}(v) \right| \right) = 0$$

P_H as a solution to a SDE

$$\begin{cases} dV_t &= -\alpha V_t dt + \sigma dB_t \\ B_t &= W_t + \int_0^t ds \int_0^s L_{Q_2}^s(s, u)(dB_u - H_u du) + \int_0^t H_s ds \end{cases}$$

P_H as a solution to a SDE

- ▶ The Law of V_0 is μ_0 .
- ▶ The law of V_t is P_H .
- ▶ $Q_2 = \mathbb{E}^H[P_H^{\otimes 2}]$
- ▶ W_t is a Wiener process under P_H .
- ▶

$$L_{Q_2}^t(s, u) = \mathbb{E}^G \left[\left\{ \frac{\exp -\frac{1}{2\sigma^2} \int_0^t G_s^2 ds}{\mathbb{E}^G \left[\exp -\frac{1}{2\sigma^2} \int_0^t G_s^2 ds \right]} \right\} G_s G_u \right]$$

where G is a centered Gaussian process with covariance $\frac{1}{2} \mathbb{E}_{Q_2}[(S(v_s^1) - S(v_s^2))(S(v_t^1) - S(v_t^2))]$.

- ▶ H is a Gaussian process with mean $\mathbb{E}_{Q_1}[S(v_t)]$ and covariance $\mathbb{E}_{Q_2}[S(v_s^1)S(v_t^2)]$.

Complete annealing

Note that the result is in general random.

Proposition

$$Q_r = Q_1^{\otimes r} \text{ iff } \int f(v_t) dQ_1(v) = 0 \quad \forall t \leq T$$

- ▶ Note that this cannot hold if f is the sigmoid between 0 and 1.
- ▶ It may hold for the sigmoid between -1 and 1 used by Crisanti, Sommers and Sompolinsky, [SCS88].
- ▶ In general for a "real" rate function (positive) the quenched limit is different from the annealed one.

Sanov's theorem

Theorem

Let V^k be an i.i.d. sequence of elements of \mathcal{T} distributed as P .





The law Π_0^N of the empirical measure under $P^{\otimes N}$ satisfies a LDP with good rate function $\mu \rightarrow I(\mu|P)$, $\mu \in \mathcal{P}(\mathcal{T})$.

Moreover, the sequence (Π_0^N) is exponentially tight:





For each $L > 0$, there exists a compact K_L of $\mathcal{P}(\mathcal{T})$ such that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \Pi_0^N(K_L^c) \leq -L$$

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